**The Derivatives Technology Foundation**

**Project Proposal**

"Skew Modeling using the SABR Model"

**Project Description**
The SABR Model for option pricing under stochastic volatility[1] is given by the following stochastic differential equations for the forward stock value f under the riskneutral measure Q:

$$df_t = \alpha_t (f_t)^{\beta} dW_t^1$$
$$d\alpha_t = v\alpha_t dW_t^2$$

where the elasticity coefficient β and the volatility of volatility v are assumed to be known constants. The two Brownian Motions are assumed to be correlated with constant correlation coefficient ρ.

The model seems to derive its popularity from the fact that the (Black-Scholes based) implied volatilities, as a function of strike K for a fixed maturity T and stock price level S, can be obtained in a series expansion which gives a more or less explicit form. Moreover, from these equations one may show that the smile dynamics generated by the model are such that for decreasing (increasing) stock price levels, the smile shifts to the left (the right). This is the behavior usually encountered in the markets, and thus a desirable feature of the model, in contrast to most local volatility models where stock prices and skew curves move in the wrong (opposite) direction of each other.

The equation for the skew in terms of the forward price f equals

$$\sigma_{imp}(f,\alpha,K) = \frac{\alpha}{f^{1-\beta}}\left(1 - \tfrac{1}{2}[1-\beta-\rho\lambda]\log\frac{K}{f} + \tfrac{1}{12}[(1-\beta)^2 + (2-3\rho^2)\lambda^2 \log^2\frac{K}{f} + ..\right)$$

where

$$\lambda = \frac{v}{\alpha} f^{1-\beta}$$

is the ratio between the volatility of volatility v and the local volatility α/f. Note that the accuracy of this expansion depends on the ratio K/f, so in principle the formula is only valid for strikes that are not too far away from the forward price. From this formula we can see that

- The ATM volatility is given by αf^{1- β} so the shape of the 'backbone' of the volatility smile (the relationship between ATM implied vol and stock price level) is controlled by the parameter β and in particular it is flat (in f) when we take β =1, corresponding to lognormal distributions (when conditioned on α).

- The second term models the skew. The part -(1/2)(1- β) log (K/f) is skew due to the fact that the local volatility αf^{1- β} is decreasing in f; it disappears when we take β =1. The second part (1/2) ρλ log (K/f) is the part that corresponds to *vanna*, the (negative) correlation between stock price levels and volatility levels. Likewise, the expression (1/12)(2-3ρ^2) λ^2 log^2 (K/f ) in the last term can be associated with *volga*, the second derivative of the price with respect to the volatility.

There exists a time-varying version of the model, where the vol of vol and correlation become time-dependent v=v(t), ρ = ρ(t), while the right-hand side of the stochastic differential equation for f is also multiplied by a time-varying function γ(t). All these time-varying functions are still chosen to be deterministic.

The main advantage of the SABR model when compared to other stochastic volatility models such as the one introduced by Heston, is the more or less explicit (asymptotic) formula for implied volatilities which makes it easier to determine the calibration parameters since they can be directly related to the shape of the smile. A case study to fit options on the South Africa Futures Exchange (SAFEX) is encouraging [2], but it seems hard to distinguish between the parameters $\nu$ (vol of vol) and $\rho$ (correlation), a problem that we encountered during earlier TDTF projects where the Heston model was used.

Since the stock dynamics are relatively easy it seems a logical first step to reproduce the smiles numerically using either Monte Carlo methods or tree-based methods (in two dimensions, see [3]). If this can be done with sufficient accuracy, we can check the series expansions mentioned in the paper and see how well options can be calibrated. At the same time, implementation of such methods would ensure that we can price American options and options with cash dividends as well.

We would like to

- design a method which is fast and accurate, to reproduce the European option prices generated by the formula for the implied volatility that is given above, based on
  - Monte Carlo Simulations, and/or
  - Two-dimensional trees
- extend such methods for cases where we have options with early-exercise opportunities (ie. American options) or options on underlying assets that pay (cash) dividends at deterministic times in the future.
- Test these methods by using them to
  - fit the SABR model to real market data
  - investigate the stability of the relevant parameters under market conditions.

**Starting Literature**


1. Hagan, P.S. & Kumar, D. & Lesniewski, A.S. & Woodward, D.E., Managing Smile Risk, *Wilmott Magazine*, pp. 84-108, 2002.
2. West, G., Calibration of the SABR Model in Illiquid Markets, *Applied Mathematical Finance,* 12(4), pp. 371-385, 2005.
3. Vellekoop, M.H. & Nieuwenhuis, J.W., A tree-based Method to price American Options in the Heston Model. *Preprint,* University of Twente, 2006.
4. Hagan, P.S. & Lesniewski, A.S. & Woodward, D.E., Probability Distribution in the SABR Model of Stochastic Volatility, Working Paper, 2004.
   Download via www.wilmott.com/attachments/SABR_ProbDistr.zip.


**Proposed Location for Project**
Saen Options

**Proposed Duration for Project**
6 to 9 months

**Supervisor**
Michel Vellekoop, TDTF & University of Twente

**Co-supervisors**
Bastiaan de Geeter, TDTF & Saen Options
Nico vd Hijligenberg, TDTF & SFISS Financial Technology

Feedback will also be sought from the other parties who participate in TDTF.


Michel Vellekoop
The Derivative Technology Foundation.

Version 27-08-07.